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Correlation Functions for Spin Systems of High Spatial Dimensionality[1]

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We investigate the correlation functions and the critical exponent ν for Ising models and spherical models on d-dimensional hypercubic lattices in the limit $d \to \infty$. Our results include a generalization of the Ornstein–Zernike theory, and an alternative explanation of the crossover phenomenon described by Baker.

KEY WORDS: Ising model; spherical model; correlation functions; spatial dimensionality; Ornstein–Zernike; critical exponent; mean-field theory.

1. INTRODUCTION

In a recent paper Baker[1] investigated the high temperature series expansion for the true range of correlation for Ising spin systems on hypercubic lattices in d dimensions. He suggested that the correlation-length critical exponent ν takes the value unity in the limit $d \to \infty$, rather than the expected classical value of one-half. Baker's suggestion appears, at first sight, to be in conflict with a theorem proved recently,[2] which states that the free energy of d-dimensional Ising systems approaches the mean-field-theory (or Curie–Weiss) value in the limit $d \to \infty$. This theory, however, only yields the classical thermodynamic exponents for specific heat, magnetization, susceptibility, etc.

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We show here that Baker's result $\nu = 1$ can *also* be obtained from a $d \to \infty$ limit.

In the theorem of Ref. 2, it was necessary to scale the nearest neighbor interaction strength by a factor d^{-1}, and take the limit $d \to \infty$ after the thermodynamic limit. Similarly, in order to discuss correlation functions for spin systems in high dimensions, a further weighting factor must be inserted before the limit $d \to \infty$ is taken. This provides a method of defining the range of correlation in the limit $d \to \infty$.

In order to motivate such a definition, consider the pair correlation function

$$U(\mathbf{0}, \mathbf{r})_{N,d} = \langle \mu_{\mathbf{0}} \mu_{\mathbf{r}} \rangle_{N,d} - \langle \mu_{\mathbf{0}} \rangle_{N,d} \langle \mu_{\mathbf{r}} \rangle_{N,d} \tag{1}$$

between two spins located at the origin and the lattice point $\mathbf{r} = (x_1, x_2, ..., x_n, 0, ..., 0)$ of an Ising system of N spins on a d-dimensional hypercubic lattice. The interaction energy $E\{\mu\}$ is given by

$$E\{\mu\} = -\frac{J}{2d} \sum_{\mathbf{r},\mathbf{r}'}^{*} \mu_{\mathbf{r}} \mu_{\mathbf{r}'} - H \sum_{\mathbf{r}} \mu_{\mathbf{r}} \tag{2}$$

for a given configuration $\{\mu\}$ of spins $\mu_{\mathbf{r}} = \pm 1$. The starred sum is over nearest neighbor lattice points $\mathbf{r}, \mathbf{r}'$. The average of any function $A\{\mu\}$ is defined by

$$\langle A \rangle = \sum_{\{\mu\}} A\{\mu\} \exp(-\beta E\{\mu\}) \Big/ \sum_{\{\mu\}} \exp(-\beta E\{\mu\}) \tag{3}$$

The combinatorial development of the Ising model (see, e.g., Ref. 3) provides a high temperature, zero field expansion of (1) in powers of

$$w = \tanh(\beta J / 2d) \tag{4}$$

The first term of this expansion corresponds to the shortest lattice walk from $\mathbf{0}$ to $\mathbf{r}$, where each step is assigned weight w, Since x_i is the total displacement in the ith direction, the weight of the complete walk is w^{Σ}, where

$$\Sigma = \sum_{i=1}^{n} x_i \qquad (x_i > 0, i = 1, 2, ..., n) \tag{5}$$

It follows from (4) that

$$w \sim (2d)^{-1} \quad \text{as} \quad d \to \infty \tag{6}$$

which implies that, for n and $\mathbf{r}$ fixed,

$$U(0, \mathbf{r})_d \sim (2d)^{-\Sigma} \quad \text{as} \quad d \to \infty \tag{7}$$

Here $U(0, \mathbf{r})_d$ is the limit as $N \to \infty$ of $U(\mathbf{0}, \mathbf{r})_{N,d}$. Therefore, to investigate the $d \to \infty$ behavior of correlation functions, we define the weighted pair correlation function $\phi(\mathbf{r})$ in the limit $d \to \infty$ by

$$\phi(\mathbf{r}) = \lim_{d \to \infty} (2d)^{\Sigma} U(\mathbf{0}, \mathbf{r})_d \tag{8}$$

To define the *range of correlation* κ, we follow Baker and, for simplicity, choose $\mathbf{r}$ to be parallel to one of the coordinate axes. This amounts to setting $n = 1$, and yields the function $\phi(x_1)$. Now we define

$$\kappa^{-1} = -\lim_{x_1 \to \infty} \frac{1}{x_1} \log|\phi(x_1)| \tag{9}$$

The critical exponent ν in infinite dimensions is then defined by

$$\nu = -\lim_{T \to T_c +} \frac{\log|\kappa|}{\log|T - T_c|} \tag{10}$$

where, as in Ref. 2 (see also Ref. 12),

$$T_c = 2J/k \tag{11}$$

which is the critical temperature in infinite dimensions. Similarly, on the low temperature side we define

$$\nu' = -\lim_{T \to T_c -} \frac{\log|\kappa|}{\log|T - T_c|} \tag{12}$$

One of our aims here is to show that ν, defined by (10), is equal to unity, in agreement with Baker's conclusions. For the Ising model we also obtain $\nu' = 1$. The implications of this result are discussed in the final section.

In outline, the derivation is arranged as follows. In the following section the theorem of Ref. 2 is generalized to a system with a nonuniform magnetic field. The result of Section 2 is used in Section 3 to evaluate the first term in the asymptotic expansion of $U(0, \mathbf{r})_d$ for large d. This term is a generalization of the Ornstein–Zernike formula, and has the same form as the exact correlation function of a finite-dimensional spherical model. The true range of correlation, defined by (9), is then determined in Section 4.

2. HIGH DIMENSIONALITY LIMIT FOR A NONUNIFORM FIELD

As a preliminary to our derivation of the formula for the pair correlation function, we here extend the theorem of Ref. 2 to include a nonuniform magnetic field.

We consider an Ising model with N spins, one at each point $\mathbf{r}$ of a d-dimensional hypercubic lattice. The total energy of a configuration $\{\mu\}$ is given by

$$E\{\mu\} = -\frac{J}{2d} \sum_{\mathbf{r},\mathbf{r}'}^{*} \mu_{\mathbf{r}}\mu_{\mathbf{r}'} - \sum_{\mathbf{r}} H_{\mathbf{r}}\mu_{\mathbf{r}} \tag{13}$$

where $H_{\mathbf{r}}$ is the field at spin site $\mathbf{r}$ and the starred sum is over nearest neighbor lattice points. We choose $H_{\mathbf{r}}$ so that it is well defined and uniformly bounded for all d; for instance, we could choose

$$H_{\mathbf{r}} = \frac{1}{d}\sum_{i=1}^{d} h(x_i) \tag{14}$$

where $\mathbf{r} = (x_1, \dots, x_d)$ and $h(x)$ is bounded.

The free energy per spin $\psi_d\{H\}$ is defined by

$$-\beta\psi_d\{H\} = \lim_{N\to\infty} N^{-1} \log Z_{N,d}\{H\} \tag{15}$$

where

$$Z_{N,d}\{H\} = \sum_{\{\mu\}} \exp(-\beta E\{\mu\}) \tag{16}$$

The main result of this section is that the limit of $-\beta\psi_d\{H\}$ as $d \to \infty$ is given by the following equivalent expressions:

$$\sup_{m\in\mathscr{I}} F\{m, H\} \tag{17}$$

$$\sup_{m\in\mathscr{I}} F'\{m, H\} \tag{18}$$

$$\lim_{N\to\infty} \max_{m\in\mathscr{I}_N} F_N\{m, H\} \tag{19}$$

$$\lim_{N\to\infty} \max_{m\in\mathscr{I}_N} F_N'\{m, H\} \tag{20}$$

where

$$F_N\{m, H\} = N^{-1} \sum_{\mathbf{r}} \{\log[2\cosh(2KM_{\mathbf{r}} + B_{\mathbf{r}})] - KM_{\mathbf{r}}m_{\mathbf{r}}\} \tag{21}$$

$$F_N'\{m, H\} = N^{-1} \sum_{\mathbf{r}} \{m_{\mathbf{r}}B_{\mathbf{r}} + KM_{\mathbf{r}}m_{\mathbf{r}} - a(m_{\mathbf{r}})\} \tag{22}$$

$$a(x) = \frac{1+x}{2}\log\left(\frac{1+x}{2}\right) + \frac{1-x}{2}\log\left(\frac{1-x}{2}\right) \tag{23}$$

$$F\{m, H\} = \lim_{N\to\infty} F_N\{m, H\} \tag{24}$$

and

$$F'\{m, H\} = \lim_{N\to\infty} F_N'\{m, H\} \tag{25}$$

$\mathscr{I}_N$ is the space of all N-tuples $(m_{\mathbf{r}_1}, \dots, m_{\mathbf{r}_N})$ where $|m_{\mathbf{r}_i}| \leqslant 1$ for all i, and $\mathscr{I}$ is the space of all sequences $(m_{\mathbf{r}_1}, m_{\mathbf{r}_2}, \dots)$, where $|m_{\mathbf{r}_i}| \leqslant 1$ for all i, which are periodic in space and are such that

$$M_{\mathbf{r}} = \lim_{d\to\infty} (2d)^{-1} \sum_{\mathbf{r}'\in(\mathbf{r})} m_{\mathbf{r}'} \tag{26}$$

exists. Here $(\mathbf{r})$ is the set of nearest neighbor points of $\mathbf{r}$. Also, $K = \beta J$ and $B_{\mathbf{r}} = \beta H_{\mathbf{r}}$, and the sums over $\mathbf{r}$ in (21) and (22) range over the N points of the lattice.

The existence of all the above limits, maxima, and suprema, and the equivalence of (17) and (19), and of (18) and (20) can be justified by arguments like those in Ref. 4, where similar results were obtained for systems with long-range (Kac) potentials and finite d.

The equivalence of (19) and (20) follows from the fact that both F_N and F_N' have their maxima for

$$m_{\mathbf{r}} = \tanh(2KM_{\mathbf{r}} + B_{\mathbf{r}}) \tag{27}$$

where they obtain the same value

$$\sum_{\mathbf{r}} \{\log 2 - KM_{\mathbf{r}}m_{\mathbf{r}} - \tfrac{1}{2}\log(1 - m_{\mathbf{r}}^2)\} \tag{28}$$

The results above reduce to those of Ref. 2 for a constant field (see Ref. 5).

The proof of our results closely follows Ref. 2. First we obtain a lower bound on $Z_{N,d}\{H\}$. Putting

$$\sigma_{\mathbf{r}} = \mu_{\mathbf{r}} - m_{\mathbf{r}} \tag{29}$$

yields

$$Z_{N,d}\{H\} = \exp\left(-K\sum_{\mathbf{r}} \bar{m}_{\mathbf{r}}m_{\mathbf{r}}\right)\prod_{\mathbf{r}}[2\cosh(2K\bar{m}_{\mathbf{r}} + B_{\mathbf{r}})] \times \left\langle \exp\left(-\frac{K}{2d}\sum_{\mathbf{r},\mathbf{r}'}^{*}\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\right)\right\rangle_c \tag{30}$$

where

$$\bar{m}_{\mathbf{r}} = (2d)^{-1}\sum_{\mathbf{r}'\in(\mathbf{r})} m_{\mathbf{r}'} \tag{31}$$

and the average $\langle\cdots\rangle_c$ is taken with respect to the distribution function

$$P\{\mu\} = \prod_{\mathbf{r}}\{\exp[(2K\bar{m}_{\mathbf{r}} + B_{\mathbf{r}})\mu_{\mathbf{r}}]\}\{2\cosh(2K\bar{m}_{\mathbf{r}} + B_{\mathbf{r}})\}^{-1} \tag{32}$$

Following the method of Ref. 2 yields

$$-\beta\psi_d\{H\} \geqslant \lim_{N\to\infty} N^{-1}\sum_{\mathbf{r}}\{\log[2\cosh(2K\bar{m}_{\mathbf{r}} + B_{\mathbf{r}})] - K\bar{m}_{\mathbf{r}}m_{\mathbf{r}}\} \tag{33}$$

for all $m_{\mathbf{r}}$ such that

$$m_{\mathbf{r}} = \tanh(2K\bar{m}_{\mathbf{r}} + B_{\mathbf{r}}) \tag{34}$$

This latter is just the condition for the sum on the right-hand side of (33) to be a maximum. Hence in the limit $d \to \infty$ we have

$$-\beta\psi\{H\} \geqslant \lim_{N\to\infty}\max_{m\in\mathscr{I}_N} F_N\{m, H\} \tag{35}$$

To show that (35) also holds if the inequality is reversed, one can use (30) and the method of Ref. 2, and show that the average $\langle\cdots\rangle_c$ in (30) is bounded above by

$$\exp(CN/d)$$

where C is a positive constant. This contributes nothing to $-\beta\psi_d\{H\}$ in the limit $d \to \infty$; so the desired results (17)–(26) are obtained.

3. THE ORNSTEIN–ZERNIKE FORMULA

The two-spin correlation function of our model may be expressed in the form

$$U(\mathbf{r}, \mathbf{r}')_{N,d} = \langle\mu_{\mathbf{r}}\mu_{\mathbf{r}'}\rangle_{N,d} - \langle\mu_{\mathbf{r}}\rangle_{N,d}\langle\mu_{\mathbf{r}'}\rangle_{N,d} = (\partial/\partial B_{\mathbf{r}})\langle\mu_{\mathbf{r}'}\rangle_{N,d} \tag{36}$$

where

$$\langle\mu_{\mathbf{r}}\rangle_{N,d} = (\partial/\partial B_{\mathbf{r}}) \log Z_{N,d} \tag{37}$$

In this section we use the formula (20) to show that the two-spin correlation function is given by the Ornstein–Zernike formula (44) for sufficiently large d. The method is similar to that used in Ref. 6.

The result (20) states that

$$\log Z_{N,d} \sim NF_N'\{m^*, H\} \tag{38}$$

as $N, d \to \infty$ ($N \gg d$) where the $m_{\mathbf{r}}^*$ maximize F_N', and hence satisfy (27). Assuming that we are justified in interchanging limits and differentiation,[7] we deduce that

$$\langle\mu_r\rangle_{N,d} \sim N\frac{\partial}{\partial B_{\mathbf{r}}} F_N'\{m^*, H\} = N\sum_{\mathbf{r}'} \frac{\partial F_N'}{\partial m_{\mathbf{r}'}^*}\frac{\partial m_{\mathbf{r}'}^*}{\partial B_{\mathbf{r}}} + N\frac{\partial F_N'}{\partial B_{\mathbf{r}}}\bigg|_{m^*} \tag{39}$$

where the $m_{\mathbf{r}}^*$ are held fixed in the final term. Since the $m_{\mathbf{r}}^*$ maximize F_N', it follows that

$$\partial F_N'/\partial m_{\mathbf{r}}^* = 0 \qquad \text{for all} \quad \mathbf{r} \tag{40}$$

Thus (22) and (39) yield

$$\langle\mu_{\mathbf{r}}\rangle_d \sim m_{\mathbf{r}}^* \quad \text{as} \quad d \to \infty \tag{41}$$

and (36) becomes

$$U(\mathbf{r}, \mathbf{r}')_d \sim (\partial/\partial B_{\mathbf{r}})m_{\mathbf{r}'}^* \quad \text{as} \quad d \to \infty \tag{42}$$

Differentiating the expression (34) for $m_{\mathbf{r}'}^*$ with respect to $B_{\mathbf{r}}$, and noting (31), then yields

$$U(\mathbf{r}, \mathbf{r}')_d \sim \{1 - (m_{\mathbf{r}'}^*)^2\}\left(\frac{K}{d}\sum_{\mathbf{r}''\in(\mathbf{r}')} U(\mathbf{r}, \mathbf{r}'')_d + \delta_{\mathbf{r}\mathbf{r}'}\right)$$

or, equivalently,

$$\frac{U(\mathbf{r}, \mathbf{r}')_d}{1 - (m_{\mathbf{r}'}^*)^2} - \frac{K}{d}\sum_{\mathbf{r}''\in(\mathbf{r}')} U(\mathbf{r}, \mathbf{r}'')_d \sim \delta_{\mathbf{r}\mathbf{r}'} \tag{43}$$

Solving (43) for $U(\mathbf{r}, \mathbf{r}')_d$ yields in the limit $N \to \infty$ (see Ref. 6)

$$U(\mathbf{0}, \mathbf{r})_d \sim \frac{1}{2K(2\pi)^d} \int_0^{2\pi} \cdots \int \frac{\exp(i\boldsymbol{\theta}\cdot\mathbf{r})}{z - \lambda(\boldsymbol{\theta})} \, d\theta_1 \cdots d\theta_d \tag{44}$$

as $d \to \infty$, where

$$z = 1/2K(1 - m^2) \tag{45}$$

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d), \qquad 0 < \theta_i < 2\pi$$

and

$$\lambda(\boldsymbol{\theta}) = \frac{1}{d} \sum_{i=1}^{d} \cos \theta_i \tag{46}$$

Here, we have set all the $B_{\mathbf{r}}$'s equal to B so that all the $m_{\mathbf{r}}{}^*$'s acquire the value m, where

$$m = \tanh(2Km + B) \tag{47}$$

The right-hand side of (44) is the general form of the Ornstein–Zernike formula. We may regard it as the leading term in an asymptotic expansion valid for large d, with correction terms of higher order in $1/d$.

We note in passing that the *direct correlation function* $C(\mathbf{r}, \mathbf{r}')$, defined as the matrix inverse of $U(\mathbf{r}, \mathbf{r}')_d$, is given from (43) by

$$C(\mathbf{r}, \mathbf{r}') \sim [1 - (m_{\mathbf{r}}{}^*)^2]^{-1} \, \delta_{\mathbf{r}\mathbf{r}'} + (K/d) \, \Delta(\mathbf{r}, \mathbf{r}') \tag{48}$$

where

$$\Delta(\mathbf{r}, \mathbf{r}') = \begin{cases} 1 & \text{if } \mathbf{r}' \in (\mathbf{r}) \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

In particular, $C(\mathbf{r}, \mathbf{r}') \sim K/d$ if $\mathbf{r}$ and $\mathbf{r}'$ are nearest neighbors. Thus, the direct correlation function is essentially the interaction potential in this limit.

It is interesting to note that the two-spin correlation function for the *spherical model* has precisely the form (44) for all d. In particular for nearest neighbor interactions of strength J/d on a d-dimensional hypercubic lattice the spherical model two spin correlation function has the form[8,9]

$$\langle s_0 s_{\mathbf{r}} \rangle = \frac{1}{2K(2\pi)^d} \int_0^{2\pi} \cdots \int \frac{\exp(i\boldsymbol{\theta}\cdot\mathbf{r})}{z - \lambda(\boldsymbol{\theta})} \, d\theta_1 \cdots d\theta_d \tag{50}$$

with $\lambda(\boldsymbol{\theta})$ defined by (46) and z determined by the saddle point condition

$$2K = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int \frac{d\theta_1 \cdots d\theta_d}{z - \lambda(\boldsymbol{\theta})} \tag{51}$$

The corresponding limiting free energy per spin is given by

$$-\beta\psi = -\tfrac{1}{2} - \tfrac{1}{2} \log 2K + Kz - \tfrac{1}{2} f(z) \tag{52}$$

where

$$f(z) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int \log\{z - \lambda(\boldsymbol{\theta})\} \, d\theta_1 \cdots d\theta_d \tag{53}$$

To investigate the $d \to \infty$ form of (50) we use the elementary identity

$$x^{-1} = \int_0^\infty e^{-tx}\, dt \tag{54}$$

and write (51) as

$$2K = \int_0^\infty e^{-tz}[I_0(t/d)]^d\, dt \tag{55}$$

Here

$$I_0(x) = (1/2\pi)\int_0^{2\pi} e^{x\cos\theta}\, d\theta \tag{56}$$

is the modified Bessel function of zeroth order.

Integrating (55) by parts, we obtain

$$2K = z^{-1}\left[1 + \int_0^\infty e^{-tz} I_1(t/d)[I_0(t/d)]^{d-1}\, dt\right] \tag{57}$$

where

$$I_1(x) = I_0'(x) \tag{58}$$

Using the inequalities ($x \geqslant 0$)

$$0 \leqslant I_1(x) \leqslant xI_0(x)/2 \leqslant xe^x|2 \tag{59}$$

then gives

$$z^{-1} \leqslant 2K \leqslant z^{-1}\left[1 + (2d)^{-1}\int_0^\infty te^{-t(z-1)}\, dt\right] \tag{60}$$

Hence, in the limit $d \to \infty$,

$$z = (2K)^{-1} \quad \text{provided} \quad 2K < 1. \tag{61}$$

When $2K \geqslant 1$ the saddle point "sticks" at $z = 1$.[8]

In a similar way one can show that $f(z)$ defined by (53) approaches $\log z$ as $d \to \infty$ ($z \geqslant 1$). It then follows from (52) that in the limit $d \to \infty$

$$-\beta\psi = \begin{cases} 0 & \text{when} \quad 2K < 1 \\ -\frac{1}{2} + K - \frac{1}{2}\log 2K & \text{when} \quad 2K \geqslant 1 \end{cases} \tag{62}$$

The form (62) agrees precisely with the corresponding expression for the Curie–Weiss spherical model.[10]

From (61) the two-spin correlation function (50) for the spherical model in the limit $d \to \infty$ is given by

$$\langle s_0 s_{\mathbf{r}} \rangle \sim \frac{1}{2K(2\pi)^d}\int_0^{2\pi}\!\!\cdots\!\!\int \frac{\exp(i\boldsymbol{\theta}\cdot\mathbf{r})}{z - \lambda(\boldsymbol{\theta})}\, d\theta_1 \cdots d\theta_d \tag{63}$$

where

$$z = \begin{cases} (2K)^{-1} & \text{for} \quad 2K < 1 \\ 1 & \text{for} \quad 2K \geqslant 1 \end{cases} \tag{64}$$

It will be noted from (44) and (45) that above the critical point ($2K < 1$ where $m = 0$) the Ising and spherical two-spin correlation functions are identical in the limit $d \to \infty$.

4. THE WEIGHTED CORRELATION FUNCTION

In this section we study the weighted correlation function

$$\phi(\mathbf{r}) = \lim_{d \to \infty} (2d)^{\Sigma} U(\mathbf{0}, \mathbf{r})_d \tag{65}$$

defined by (8), with $\mathbf{r} = (x_1, x_2, ..., x_n, 0, ..., 0)$, $x_i > 0$, and $\Sigma = \sum_{i=1}^{n} x_i$.

Our method takes (44) [or (53) for the spherical model] as a starting point. It is therefore rigorously correct for the spherical model, but relies on the assumption leading to (43) in the case of the Ising model.

Using the identity (54) we can write (44) in the form

$$U(\mathbf{0}, \mathbf{r})_d \sim \frac{1}{2K(2\pi)^d} \int_0^\infty dt\, e^{-zt} \int \overset{\pi 2}{\underset{0}{\cdots}} \int d\theta_1 \cdots d\theta_d \times \exp\left(i\boldsymbol{\theta}\cdot\mathbf{r} + \frac{t}{d} \sum_{i=1}^{d} \cos \theta_i \right) \tag{66}$$

Using the identity

$$I_x(y) = (1/2\pi) \int_0^{2\pi} \exp(i\theta x + y \cos \theta)\, d\theta \tag{67}$$

for the modified Bessel function of the first kind of order x yields

$$U(\mathbf{0}, \mathbf{r})_d \sim \frac{1}{2k} \int_0^\infty e^{-zt} I_0\left(\frac{t}{d}\right)^{d-n} \prod_{j=1}^{n} I_{x_j}\left(\frac{t}{d}\right) dt \tag{68}$$

for $\mathbf{r}$ with nonnegative components x_i where $x_i = 0$ for $n < i \leqslant d$.

From the series expansion (for integral x)

$$I_x(y) = \sum_{s=0}^{\infty} \frac{(y/2)^{x+2s}}{(x+s)!\, s!} \tag{69}$$

we note that $I_0(y) \geqslant 1$ ($y \geqslant 0$) and that

$$\begin{aligned} I_0(y)^k - 1 &= (I_0 - 1)(I_0^{k-1} + I_0^{k-2} + \cdots + 1) \\ &\leqslant (I_0 - 1) k I_0^{k-1} \\ &\leqslant (I_0 - 1) k e^{(k-1)y} \end{aligned} \tag{70}$$

since (69) implies $I_0(y) \leqslant e^y$. Again (69) gives

$$I_0(y) - 1 = \left(\frac{y}{2}\right)^2 \sum_{x=0}^{\infty} \frac{(y/2)^{2s}}{(s+1)!} \leqslant \frac{y^2}{4} e^y$$

so that from (70)

$$1 \leqslant I_0(y)^k \leqslant 1 + k\frac{y^2}{4} e^{ky} \tag{71}$$

Similarly, (69) gives

$$\frac{(y/2)^x}{x!} \leqslant I_x(y) \leqslant \frac{(y/2)^x e^y}{x!} \tag{72}$$

which yields

$$\frac{(y/2)^\Sigma}{\prod x_j!} \leqslant \prod_{j=1}^{n} I_{x_j}(y) \leqslant \frac{(y/2)^\Sigma e^{ny}}{\prod x_j!} \leqslant \frac{(y/2)^\Sigma}{\prod x_j!}(1 + nye^{ny}) \tag{73}$$

where $\Sigma = \sum_{i=1}^{n} x_i$.

Substituting (71) and (73) into (68) gives

$$U(\mathbf{0}, \mathbf{r})_d \sim \frac{1}{2K}\int_0^\infty dt\, e^{-zt} \prod_{j=1}^{n} \frac{(t/2d)^{x_j}}{x_j!} + \epsilon$$

$$= \frac{1}{2Kz}\left(\frac{\Sigma!}{\prod x_j!}\right)\left(\frac{1}{2dz}\right)^\Sigma + \epsilon \tag{74}$$

where

$$0 \leqslant \epsilon \leqslant \frac{1}{2K}\int_0^\infty dt\, e^{-zt} \frac{(t/2d)^\Sigma}{\prod x_j!}$$

$$\times \left\{\frac{nt}{d} e^{nt/d} + (d-n)\left(\frac{t}{2d}\right)^2 e^{(d-n)t/d} + \frac{n(d-n)}{4}\left(\frac{t}{d}\right)^3 e^t\right\}$$

$$\leqslant \frac{1}{2K(2d)^{\Sigma+1}}\int_0^\infty dt\, e^{-(z-1)t}(t^{\Sigma+1} + \tfrac{1}{4}t^{\Sigma+2} + \tfrac{1}{4}t^{\Sigma+3})$$

$$= O((2d)^{-\Sigma-1}) \qquad \text{for} \quad z > 1 \tag{75}$$

The definition (65) applied to (74) therefore yields

$$\phi(\mathbf{r}) = \frac{1}{2Kz}\left(\frac{\Sigma!}{\prod x_j!}\right) z^{-\Sigma} \qquad \text{provided} \quad z > 1 \tag{76}$$

This formula is essentially the first term in the high temperature expansion of $U(\mathbf{0}, \mathbf{r})_d$. The combinatorial factor is simply the total number of walks of minimum length connecting 0 and $\mathbf{r}$.

To determine the range of correlation (9), we first take $n = 1$ in (76) and obtain

$$\phi(x_1) = (1/2Kz)z^{-x_1} \tag{77}$$

Then (9) immediately gives

$$\kappa^{-1} = \log|z| \tag{78}$$

Noting that the slope of the curve $y = \tanh(2Km + B)$ is strictly less than unity where it intersects $y = m\ (>0)$, we have for all K and $B > 0$ that, for the Ising model,

$$z = 1/2K(1 - m^2) > 1 \tag{79}$$

and moreover for $B = 0$

$$z - 1 \sim |T - T_c| \quad \text{as} \quad T \to T_c \quad (2K \to 1) \tag{80}$$

It then follows from (10) and (12) that for the Ising model in infinite dimensions $\nu = \nu' = 1$.

For the spherical model, (78) also holds with z given by (54). It follows that κ is only defined for $T > T_c$ ($z = 1$ for $T \leqslant T_c$) and from (1.10) that $\nu = 1$.

Baker's result is thus established for both models.

5. DISCUSSION

The main new result of this paper is the generalized Ornstein–Zernike formula (44) for an Ising model of high dimensionality. A corresponding formula for three-spin correlations, like that of Ref. 6, can be obtained by the same method.

The question of the value of ν can be resolved as follows. Its value ν_d for a d-dimensional Ising model is defined by

$$\nu_d = -\lim_{T \to T_{c,d}} \frac{\log|\kappa_d|}{\log|T - T_{c,d}|} \tag{81}$$

where $T_{c,d}$ is the critical temperature and

$$\kappa_d - 1 = -\lim_{|\mathbf{r}| \to \infty} \frac{1}{|\mathbf{r}|} \log U(\mathbf{0}, \mathbf{r})_d \tag{82}$$

For the spherical model, it is known[9] that

$$\nu_d = \tfrac{1}{2} \qquad \text{for} \quad d \geqslant 3 \tag{83}$$

Since the Ornstein–Zernike formula (44) for the Ising model agrees precisely with the spherical model result for large d, one might expect that

$$\lim_{d \to \infty} \nu_d = \tfrac{1}{2} \tag{84}$$

for the Ising model. This is also supported by renormalization group arguments,[11] which suggest that $\nu_d = \frac{1}{2}$ for $d \geqslant 4$.

Comparing (9) and (10) with (81) and (84), we see that there is no conflict between (84) and the result $\nu = 1$. In the former, the limit $d \to \infty$ is taken *first* (with a weighting factor), while in the latter case, it is taken *last*. The different results follow simply from taking limits in a different order. This fact is reflected in Baker's analysis of the series for κ_d^{-1}: Depending on how the series are analyzed, it is possible to obtain either $\nu = \frac{1}{2}$ or $\nu = 1$.

Finally, we note that to extend the definition (9) to an arbitrary vector $\mathbf{r} = r\mathbf{e}$, the combinatorial factor in (76) must be absorbed into the definition (8) of $\phi(\mathbf{r})$, and $\lim_{x_1 \to \infty} x_1^{-1} \cdots$ replaced by $\lim_{\Sigma \to \infty} \Sigma^{-1} \cdots$. With a κ_e^{-1} so defined, it follows from (10) and (76) that $\nu = 1$ as before.

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